Galilean covariant effective theory for bound states of heavy mesons

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Abstract

In this work we formulate the Galilei-covariant version of an effective theory containing non-relativistic heavy mesons and pions as degrees of freedom. This manifestly Galilean covariant framework is based on a five-dimensional space-time that has been used in the description of covariant non-relativistic physics. In this context, effective Lagrangian is introduced without ambiguities, containing kinetic and interaction terms that are naturally Galilean invariant. The leading-order scattering amplitudes and the properties of possible heavy-meson bound states are calculated and discussed.

Keywords: effective field theories; charm mesons; Galilean covariance

1 Introduction

More than four decades ago, the idea of hadron molecular state was proposed [1], and since then it has been employed for the description of deuteron-like meson-meson bound states [2,3]. But it became, in fact, a hot research topic after the discovery of exotic hadron states (denoted by X, Y and Z states) in 2003 [4,5]. The reason is due to the proximity of the X, Y and Z masses to some hadronic thresholds, which allows us to understand them as heavy-meson bound states if they are below the threshold and in the first Riemann sheet of transition amplitude. Taking as example the famous $X(3872)$ state, it might be interpreted as a loosely bound state of charmed mesons, i.e. $(D\overline{D}^* + c.c.)$ [5–25]. Other exotic states are also considered in the molecular interpretation; see for instance discussions in Refs. [5,24–28].

Motivated by the feature that in heavy-hadron phenomenology at low energies their masses are much larger than their momenta, several works reported in literature have investigated the heavy-meson dynamics via non-relativistic effective theories [7–9,13–16,20–22,29–32]. In particular, Fleming et al. have developed an effective theory of non-relativistic charm mesons (D and D^*) and pions (π) that can be used to compute the properties of the state $X(3872)$ at low energies [9]. In this so-called XEFT (which is the acronym for '$X(3872)$ effective field theory'), the fundamental degrees of freedom D and D^* might interact through pion-exchange or four-body coupling [9].

The XEFT has underwent some improvements in order to obtain accurate quantitative predictions [8,13,14,20,21]. As pointed out in Ref. [20], some of these problems are due to its formulation as a non-relativistic field theory of the charm mesons and pions that is not Galilean invariant. In principle, Galilean invariance is the underlying symmetry of low-energy systems encountered in condensed matter physics, nuclear physics, and the like. It is therefore natural to find it also in low-energy effective approaches to subatomic physics, such as XEFT.

On the other hand, there are consistent examples which explore manifestly Galilean covariant versions of non-relativistic field theories [33–42]. They are based on a five-dimensional space used to construct a covariant non-relativistic physics. In this sense, manifestly Galilei-covariant wave equation written similarly to Klein-Gordon equation can be associated to the covariant version of the Schrödinger equation.

Thus, our purpose is to use the manifestly covariant formalism of Galilean invariance [33,34], henceforth named simply 'Galilean covariance', to formulate the XEFT without ambiguities. We also analyze the leading-order scattering amplitudes and the properties of possible heavy-meson bound states.

The paper is organized as follows. Sec. 2 presents an overview of the Galilei-covariant formalism. In Sec. 3, we establish a Galilean covariant approach to XEFT, and apply it to the scattering of the charm mesons $D^*\overline{D} \to D^*\overline{D}$ in Sec. 4. In Sec. 5, we examine bound states. Finally, Sec. 6 contains our concluding remarks.

2 Galilean covariance

The formalism of Galilean covariance utilized hereafter is motivated by the fact that the Galilei group in $3+1$ space-time is a subgroup of the Poincaré group in $4+1$ space-time. Therefore, Galilean covariance is similar to the Lorentz-covariant approach of relativistic field theory. Galilean covariance is based on an equivalent metric, called 'Galilean metric',

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{1}$$

such that the scalar product of so-called 'Galilean five-vectors' v and w is

$$(v|w) = \eta_{\mu\nu} v^\mu w^\nu = \sum_{i=1}^{3} v^i w^i - v^4 w^5 - v^5 w^4. \tag{2}$$

In analogy with Lorentz and Poincaré covariance, the scalar product of infinitesimal elements of Galilean five-coordinates is invariant under the inhomogeneous Galilean transformations (with relative velocity $\mathbf{V}$) :

$$\overline{x}^j = R^j_k x^k + V^j x^4 + a^j, \qquad j,k = 1,2,3, \tag{3}$$

$$\overline{x}^4 = x^4 + a^4, \tag{4}$$

$$\overline{x}^5 = x^5 + V_j R^j_k x^k + \frac{1}{2}\mathbf{V}^2 x^4 + a^5. \tag{5}$$

For future convenience, let us mention that the corresponding transformations for the derivatives imply the following invariant expression,

$$f\left(\overleftarrow{\partial}_5 \overrightarrow{\nabla} - \overrightarrow{\partial}_5 \overleftarrow{\nabla}\right) g, \tag{6}$$

with f and g functions of the five-coordinates, and where the arrows denote the left or right derivatives.

A simple motivation for Eq. (5) follows from observing that under the action of a Galilean boost on momentum (in one dimension),

$$\overline{p} = m(v - V) = p - mV, \tag{7}$$

then the energy transforms as

$$\overline{E} = \frac{\overline{p}^2}{2m} = \frac{1}{2m}(p - mV)^2 = E - pV + \frac{1}{2}mV^2. \tag{8}$$

This expression is analogous to Eq. (5) with x^k, x^4 and x^5 corresponding to p, m and E, respectively. Hence, a general Galilean transformation of five-vectors x^μ shown in Eqs. (3)-(5) can therefore be written in the form

$$\overline{x}^\mu = G^\mu_\nu x^\nu + a^\mu, \tag{9}$$

with

$$G^\mu_\nu = \begin{pmatrix} R^1_1 & R^1_2 & R^1_3 & v^1 & 0 \\ R^2_1 & R^2_2 & R^2_3 & v^2 & 0 \\ R^3_1 & R^3_2 & R^3_3 & v^3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v_i R^i_1 & v_i R^i_2 & v_i R^i_3 & \frac{1}{2}\mathbf{v}^2 & 1 \end{pmatrix}, \tag{10}$$

where R^j_k are rotation matrices.

Representations of the group constituted by the transformations in Eq. (9) in Hilbert space allows us to obtain Galilei-covariant fields. In this sense, from the Casimir invariants, we get the following equations for the scalar representation [33–37, 39–42]:

$$\left(\partial_\mu \partial^\mu - k^2\right) \Phi(x) = 0, \tag{11}$$

and

$$\partial_5 \Phi(x) = -im\Phi(x), \tag{12}$$

where k and m are parameters related to the invariants, with m being associated to the mass of the quanta associated with this field. Eq. (12) implies that Φ may be written in the form

$$\Phi(x) = e^{-imx^5}\psi(\mathbf{x}, t), \tag{13}$$

so that Eq. (11) becomes the free Schrödinger equation,

$$\left(-\frac{\nabla^2}{2m}+\frac{k^2}{2m}\right)\psi(\mathbf{x},t)=i\partial_t\psi(\mathbf{x},t). \tag{14}$$

with $\psi(\mathbf{x},t)$ being the Schrödinger field. The term $\frac{k^2}{2m}$ is a constant that can be added to energy. The extra coordinate x^5 is a real number, and we will interpret any integral over it as

$$\int dx^5 \rightarrow \lim_{l\rightarrow\infty}\frac{1}{2l}\int_{-l}^{l}dx^5. \tag{15}$$

Then, an integral over x^5 will be reduced to the usual integral over $(3,1)$ spacetime if the integrand is independent of x^5. This will prevent us from carrying along the factor l.

The expression in Eq. (11) can be understood as a $(4+1)$ "Galilean Klein-Gordon-Fock equation". Thus, we will employ the Galilei-covariant fields of type $\Phi(x)$ to build the Galilean covariant version of XEFT.

3 Galilean covariant formulation of XEFT

The XEFT Lagrangian consists of the non-relativistic fields for charmed mesons and pions. Hereafter, we represent the π^0 pion field by π, and we represent by D and $\mathbf{D}$ the pseudo-scalar field and vector field, respectively, related to mesons D^0 and D^{*0}, where the superscript 0 implies that these mesons have neutral electric charge. We follow Ref. [20] and consider the kinetic terms for the Galilean invariant XEFT Lagrangian:

$$\mathcal{L}_\pi = \pi^\dagger\left(i\partial_t+\frac{\nabla^2}{2m_\pi}\right)\pi, \tag{16}$$

$$\mathcal{L}_D = D^\dagger\left(i\partial_t+\frac{\nabla^2}{2m_D}\right)D, \tag{17}$$

$$\mathcal{L}_\mathbf{D} = \mathbf{D}^\dagger\cdot\left(i\partial_t+\frac{\nabla^2}{2m_{D^*}}\right)\mathbf{D}-\delta\mathbf{D}^\dagger\cdot\mathbf{D}, \tag{18}$$

where the masses of π^0, D^0 and D^{*0} are denoted by m_π, m_D and m_{D^*}, respectively, and

$$\delta = m_{D^*}-m_D-m_\pi. \tag{19}$$

The kinetic terms for the mesonic antiparticles $\overline{D}$ and $\overline{\mathbf{D}}$ have similar expressions,

$$\mathcal{L}_{\overline{D}} = \overline{D}^{\dagger}\left(i\partial_t + \frac{\nabla^2}{2m_D}\right)\overline{D}, \tag{20}$$

$$\mathcal{L}_{\overline{\mathbf{D}}} = \overline{\mathbf{D}}^{\dagger}\cdot\left(i\partial_t + \frac{\nabla^2}{2(m_{D^*})}\right)\overline{\mathbf{D}} - (\delta)\,\overline{\mathbf{D}}^{\dagger}\cdot\overline{\mathbf{D}}. \tag{21}$$

The Galilean invariant XEFT Lagrangian contains other terms which represent the $D^*D\pi$ and four-body interactions. The interaction Lagrangian related to transition $D^{*0} \leftrightarrow D^0\pi^0$ is written as

$$\begin{aligned}\mathcal{L}_{D^*\leftrightarrow D\pi} &= \frac{g}{2\sqrt{m_\pi}f_\pi(m_{D^*})}\left\{\mathbf{D}^{\dagger}\cdot\left[D\left(m_D\overrightarrow{\nabla} - m_\pi\overleftarrow{\nabla}\right)\pi\right]\right.\\ &\quad\left.+\left[D\left(m_D\overrightarrow{\nabla} - m_\pi\overleftarrow{\nabla}\right)\pi\right]^{\dagger}\cdot\mathbf{D}\right\},\end{aligned} \tag{22}$$

where $\overleftarrow{\nabla}$ and $\overrightarrow{\nabla}$ denote the left- and right-gradients, respectively, and $f_\pi \approx 130$ MeV is the pion's decay constant. Similarly, the XEFT interaction Lagrangian for the transition $\overline{D}^{*0} \leftrightarrow \overline{D}^0\pi^0$ is obtained by replacing in Eq. (22) the fields $\mathbf{D}$ and D by $\overline{\mathbf{D}}$ and $\overline{D}$, respectively. On the other hand, the interaction Lagrangian for the transition $D^{*0}\overline{D}^0 \to D^{*0}\overline{D}^0$ contains the contact and ∇^2 terms; that is,

$$\begin{aligned}\mathcal{L}_{D^*\overline{D}\to D^*\overline{D}} &= -C_1\left(\overline{D}\mathbf{D}\right)^{\dagger}\cdot\left(\overline{D}\mathbf{D}\right)\\ &\quad+\frac{C}{4(2m_{D^*})^2}\left\{\left(\overline{D}\mathbf{D}\right)^{\dagger}\cdot\left[\overline{D}\left(m_D\overrightarrow{\nabla} - (m_{D^*})\overleftarrow{\nabla}\right)^2\mathbf{D}\right]\right.\\ &\quad\left.+\left[\overline{D}\left(m_D\overrightarrow{\nabla} - m_{D^*}\overleftarrow{\nabla}\right)^2\mathbf{D}\right]^{\dagger}\cdot\left(\overline{D}\mathbf{D}\right)\right\}.\end{aligned} \tag{23}$$

In order to apply Galilean covariance to the XEFT, we utilize Eq. (13) and express the fields $\pi^0(\mathbf{x},t)$, $D^0(\mathbf{x},t)$, $D^{*0}(\mathbf{x},t)$, defined in the $3+1$ spacetime, as fields in the five-dimensional Galilean manifold:

$$\begin{aligned}\pi(x) &= e^{-im_\pi x^5}\pi^0(\mathbf{x},t),\\ D(x) &= e^{-im_D x^5}D^0(\mathbf{x},t),\\ D_\mu(x) &= e^{-im_{D^*}x^5}\quad D^{*0}(\mathbf{x},t),\end{aligned} \tag{24}$$

where $x = (\mathbf{x}, x^4, x^5)$ and $D_\mu = (\mathbf{D},\ 0,\ 0)$. Similarly we rewrite for the fields $\overline{D}^0(\mathbf{x},t)$ and $\overline{D}^{*0}(\mathbf{x},t)$ the expressions $\overline{D}(x) = e^{-im_D x^5}\overline{D}^0(\mathbf{x},t)$ and

$\overline{D}_\mu(x) = e^{-im_{D^*}x^5}\overline{D}^{*0}(\mathbf{x},t)$. Then, the free Lagrangians in Eqs. (16)-(18) are written in manifestly covariant formalism as

$$\begin{aligned}
\mathcal{L}_\pi &= \frac{1}{2m_\pi}(\partial_\mu\pi)^\dagger(\partial^\mu\pi),\\
\mathcal{L}_D &= \frac{1}{2m_D}(\partial_\mu D)^\dagger(\partial^\mu D),\\
\mathcal{L}_{\overline{D}} &= \frac{1}{2m_D}(\partial_\mu\overline{D})^\dagger(\partial^\mu\overline{D}),\\
\mathcal{L}_{D^*} &= \frac{1}{2m_{D^*}}(\partial_\mu D_\nu)^\dagger(\partial^\mu D^s\nu) - \delta D_\mu^\dagger D^\mu,\\
\mathcal{L}_{\overline{D}^{*0}} &= \frac{1}{2m_{D^*}}\left(\partial_\mu\overline{D}_\nu\right)^\dagger(\partial^\mu\overline{D}^\mu) - \delta\overline{D}_\mu^\dagger\overline{D}^\mu,
\end{aligned} \tag{25}$$

with δ defined in Eq. (19) and with the Galilean scalar product prescribed by Eq. (2). The corresponding equations of motion obtained from these Lagrangians are

$$\partial_\mu\partial^\mu\pi = 0, \tag{26}$$

$$\partial_\mu\partial^\mu D = 0, \tag{27}$$

$$\left(\partial_\mu\partial^\mu + 2m_{D^*}\delta\right)D_\mu = 0. \tag{28}$$

They have the form of the covariant free wave equation in Eq. (11), as expected since they involve quantities that behave as Schrödinger fields.

Now we can discuss the interactions. The interaction Lagrangian for the transition $D^{*0} \leftrightarrow D^0\pi^0$ in Eq. (22) contains terms like $\mathbf{D}^\dagger D\nabla\pi^0$, $\left(\overline{D}\mathbf{D}\right)^\dagger \overleftrightarrow{\nabla}\mathbf{D}$ which are not Galilei invariant [20]. In order to solve this problem, we use same idea as in Ref. [20]: we replace an expression like $f\nabla g$ by its Galilean covariant form from Eq. (6): $\frac{1}{m_f+m_g} f\left(\overleftarrow{\partial}_5\overrightarrow{\nabla} - \overrightarrow{\partial}_5\overleftarrow{\nabla}\right)g = \frac{1}{m_f+m_g} f\left(m_f\overrightarrow{\nabla} - m_g\overleftarrow{\nabla}\right)g$. Thus, the pion interaction terms for the transition $D^{*0} \leftrightarrow D^0\pi^0$ are

$$\begin{aligned}
\mathcal{L}_{D^*\leftrightarrow D\pi} &= \frac{g}{2\sqrt{m_\pi}f_\pi m_{D^*}}\left\{D_\mu^\dagger\left[D\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\mu - \overrightarrow{\partial}_5\overleftarrow{\partial}^\mu\right)\pi\right]\right.\\
&\quad\left. + \left[D\left(\overleftarrow{\partial}_5\overrightarrow{\partial}_\mu - \overrightarrow{\partial}_5\overleftarrow{\partial}_\mu\right)\pi\right]^\dagger D^\mu\right\},
\end{aligned} \tag{29}$$

where we have used the mass conservation, that is $m_{D^*} = m_D + m_\pi$. Similarly, the terms related to pions interaction for the transition $\overline{D}^{*0} \leftrightarrow \overline{D}^0\pi^0$ are obtained by substituting $\mathbf{D}$ and D by $\overline{\mathbf{D}}$ and $\overline{D}$, respectively.

In the case of the interaction Lagrangian for the transition $D^{*0}\overline{D}^0 \to D^{*0}\overline{D}^0$ in Eq. (23), the manifestly covariant version of $\mathcal{L}_{D^*\overline{D}\to D^*\overline{D}}$ is given by:

$$
\begin{aligned}
\mathcal{L}_{D^*\overline{D}\to D^*\overline{D}} &= -C_1\overline{D}^\dagger\overline{D}D^\dagger_\mu D^\mu \\
&+\frac{C_3}{4(2m_{D^*})^2}\left\{\left(\overline{D}D_\mu\right)^\dagger\left[\overline{D}\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2D^\mu\right]\right. \\
&\left.+\left[\overline{D}\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2D^\mu\right]^\dagger\overline{D}D_\mu\right\}. \qquad (30)
\end{aligned}
$$

The covariant XEFT Lagrangian in Eq. (30) can be generalized by adding other interactions. Inspired by Ref. [7], we can also consider the four-body interactions $D\overline{D}^* \to D\overline{D}^*$, $D^*\overline{D} \to D\overline{D}^*$ and $D\overline{D}^* \to D^*\overline{D}$ (without 0 superscripts)

$$
\begin{aligned}
\mathcal{L}_{D\overline{D}^*\to D\overline{D}^*} &= -C_1\overline{D}^\dagger_\mu\overline{D}^\mu D^\dagger D \\
&+\frac{C_3}{4(2m_{D^*})^2}\left\{\left(D\overline{D}_\mu\right)^\dagger\left[D\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2\overline{D}^\mu\right]\right. \\
&\left.+\left[D\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2\overline{D}^\mu\right]^\dagger D\overline{D}_\mu\right\}, \\
\mathcal{L}_{D^*\overline{D}\to D\overline{D}^*} &= C_2\overline{D}^\dagger_\mu D^\mu\overline{D}D^\dagger \\
&+\frac{C_4}{4(2m_{D^*})^2}\left\{\left(D\overline{D}_\mu\right)^\dagger\left[\overline{D}\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2D^\mu\right]\right. \\
&\left.+\left[\overline{D}\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2D^\mu\right]^\dagger D\overline{D}_\mu\right\}, \\
\mathcal{L}_{D\overline{D}^*\to D^*\overline{D}} &= C_2D^\dagger_\mu\overline{D}^\mu D\overline{D}^\dagger \\
&+\frac{C_4}{4(2m_{D^*})^2}\left\{\left(\overline{D}D_\mu\right)^\dagger\left[D\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2\overline{D}^\mu\right]\right. \\
&\left.+\left[D\left(\overleftarrow{\partial}_5\overrightarrow{\partial}^\nu-\overrightarrow{\partial}_5\overleftarrow{\partial}^\nu\right)^2\overline{D}^\mu\right]^\dagger\overline{D}D_\mu\right\}. \qquad (31)
\end{aligned}
$$

As a consequence, we see that the Lagrangians given in Eqs. (25), (29), (30) and (31) are manifestly invariant under the general Galilean transformations written in Eq. (9).

The Galilean covariant propagators for the π^0, D^0 and D^{*0} are given in Ref. [38]

$$\Delta_\pi(p) = \frac{2p_5}{p_\mu p^\mu + i\epsilon}, \qquad (32)$$

$$\Delta_D(p) = \frac{2p_5}{p_\mu p^\mu + i\epsilon}, \tag{33}$$

$$\Delta^{\mu\nu}_{D^{*0}}(p) = \frac{2p_5 g^{\mu\nu}}{p_\mu p^\mu - k - i\epsilon}, \tag{34}$$

where $k = 2m_{D^*}\delta$. The propagators for $\overline{D}^0$ and $\overline{D}^{*0}$ are represented in the same way.

The Feynman rules for interactions involving pions engendered by Eq. (29) can be written in the covariant formalism by

$$\frac{g}{2\sqrt{m_\pi}f_\pi}\frac{(p_5 q_\mu - q_5 p_\mu)}{m_{D^*}}, \tag{35}$$

where q^μ and p^μ are outgoing five-momenta for π^0 and the charm mesons, respectively.

Similarly, the Feynman rules for the four-body interactions $D^*\overline{D} \to D^*\overline{D}$ in Eq. (30) are given by

$$-iC_1 - i\frac{C_3}{4}\frac{[(m_{D^*})p - m_D p_*]^2 + [(m_{D^*})p' - m_D p'_*]^2}{(2m_{D^*})^2}, \tag{36}$$

where p and p_* are the five-momenta for incoming spin-zero and spin-one charm mesons, and p' and p'_* are the five-momenta for outgoing mesons. The other four-body interactions in Eq. (30) must be treated in the same manner.

4 The $D^*\overline{D} \to D^*\overline{D}$ scattering

In this section, we analyze the scattering amplitude for $2 \to 2$ processes, taking the reaction $D^*\overline{D} \to D^*\overline{D}$ as one example that can be directly generalized to other interactions engendered by the Lagrangians introduced in the previous section. In the present approach, we consider the process at leading order in XEFT, which means that we ignore the pion interactions and also neglect the C_3 and C_4-dependent terms in Eqs. (29), (30) and (31). The reason is due to the fact that in the scenario of heavy hadronic molecules, pion-exchange effects are in general perturbative over the expected validity range of heavy-quark effective theories and are suppressed, as pointed out in Refs. [15, 17, 26]. Therefore, at lowest order, the effective theory can be considered as a contact-range theory, taking into account the proper range of binding energies. Thus, we explore the leading order only with contact interactions, and investigate the region where the pion-exchange contribution is not relevant. It is worth noticing that this choice follows a scenario similar to Ref. [43], which is based on an heavy-quark symmetric effective approach.

In this section, we adopt a simpler situation. Here, we apply the perturbation theory by taking into account only the first term in Eq. (30); we disregard the next-to-leading order contributions that might be yielded from the other Lagrangians in Eqs. (29) and (31). Our main purpose is to show the consistency of the Galilean covariant version of XEFT in the computation of simplest transition amplitudes. We postpone to the next section the analysis of coupled channel amplitudes.

Accordingly, the lowest-order contributions of the amplitude for the reaction $D^*\overline{D} \to D^*\overline{D}$ comes from Eq. (30) without derivatives terms,

$$\mathcal{L}_{D^*\overline{D}\to D^*\overline{D}} = -C_1 \overline{D}^\dagger \overline{D} \widetilde{D}^\dagger_\mu D^\mu . \tag{37}$$

In this sense, the scattering amplitude of the reaction $D^*\overline{D} \to D^*\overline{D}$ is obtained from the S-matrix,

$$\begin{aligned} S_{fi} &= \left\langle D^*\overline{D}\right| \mathbf{S} \left|D^*\overline{D}\right\rangle \\ &= \left\langle D\overline{D}\right| T \left\{ \exp[-i \int C_1 \overline{D}^\dagger \overline{D} D^{*\dagger}_\mu D^{*\mu} d^5x] \right\} \left|D\overline{D}\right\rangle . \end{aligned} \tag{38}$$

The lowest non-trivial contribution is obtained by expanding Eq. (38), yielding

$$S^{(1)}_{fi} = \left\langle D^*\overline{D}\right| T \left\{ -i \int C_1 \overline{D}^\dagger \overline{D} D^{*\dagger}_\mu D^{*\mu} d^5x \right\} \left|D^*\overline{D}\right\rangle . \tag{39}$$

The time-ordered product between the field in the equation above can be simplified by the Wick theorem if we consider that the molecular states $\left| D^*\overline{D}\right\rangle$ are obtained from the vacuum state $|0\rangle$ by

$$a^\dagger_{D^*} a^\dagger_{\overline{D}} |0\rangle \sim \left| D^*\overline{D}\right\rangle , \tag{40}$$

$$\langle 0|\, a_{D^*} a_{\overline{D}} \sim \left\langle D^*\overline{D}\right| , \tag{41}$$

where $a^\dagger_{D^*}$ ($a^\dagger_{\overline{D}}$) and a_{D^*} ($a_{\overline{D}}$) are the creation and annihilation operators for D^* ($\overline{D}$).

If we apply the usual techniques and introduce the Galilean invariant scattering matrix $\mathbf{M}$ at order n as [37, 39],

$$T_{fi}^{(n)} \equiv S_{fi}^{(n)} - 1 = (2\pi)^5 \delta^{(5)} \left(\sum_i p_i - \sum_j q_j \right) i\mathbf{M}^{(n)}, \tag{42}$$

with T_{fi} being the transition amplitude, then the usual invariant scattering matrix $\mathcal{M}^{(n)}$ at order n is obtained from the relation,

$$\mathbf{M}^{(n)} = [2\pi\delta(0)]^{n-1} \mathcal{M}^{(n)}. \tag{43}$$

Consequently, from Eqs. (42) and (43) in Eq. (39), we obtain the lowest-order invariant scattering matrix,

$$\mathcal{M}^{(1)}_{D^*\overline{D}} = -C_1. \tag{44}$$

If we proceed in the same way as for $S_{fi}^{(1)}$, the second-order term $S_{fi}^{(2)}$ obtained is written as one-loop contribution,

$$S_{fi}^{(2)} = -\frac{{C_1}^2}{2} \int d^5x' \int d^5x'' e^{i[(p_1'+p_2')x''-(p_1+p_2)x']} \Delta_{D^*}(x''-x')\Delta_{\overline{D}}(x''-x'), \tag{45}$$

where $\Delta_{\overline{D}}$ and Δ_{D^*} are the Galilean propagators for the pseudo-scalar meson $\overline{D}$ and vector meson D^*, respectively.

With the help of the Galilei-covariant Fourier transform of the propagators, given by [37, 39]

$$\Delta(x-y) = \frac{1}{(2\pi)^5} \int d^5k \, \tilde{\Delta}_F(k) \exp\left[ik \cdot (x-y)\right] 2\pi\delta\left(k^4 - m\right), \tag{46}$$

and the properties of the delta function, Eq. (45) becomes

$$S_{fi}^{(2)} = -2{C_1}^2 \delta^{(5)}\left(p_1 + p_2 - q_1 - q_2\right) \int \frac{d^5k}{(2\pi)^5} \, G_{DD^*} \, \overline{\delta}, \tag{47}$$

where $\overline{\delta} = 2\pi\delta(k^4 - m_{D^*})2\pi\delta(p_1^4 + p_2^4 - k^4 - m_{\overline{D}})$, and G_{DD^*} represents the loop function in momentum space defined by

$$G_{DD^*} \equiv \frac{1}{4} \widetilde{\Delta}_{D^*}(k) \widetilde{\Delta}_{\overline{D}}(p_1 + p_2 - k). \tag{48}$$

As a result, the next-to-leading order contribution for the invariant matrix is

$$\mathcal{M}^{(2)}_{D^*\overline{D}} = 2iC_1{}^2 \int \frac{d^5k}{(2\pi)^5}\, G_{DD^*}\, \widetilde{\delta}. \tag{49}$$

where $\widetilde{\delta} = \overline{\delta}/[2\pi\delta(0)]$. With the Feynman propagators in Eqs. (33) and (34) related to the fields $\overline{D}$ and D^*, and noticing that

$$\begin{aligned} k_\mu k^\mu &= \mathbf{k}^2 - 2k_4k_5, \\ (p_1 + p_2 - k)^2 &= \mathbf{k}^2 - 2m_D\left(\frac{\mathbf{p}^2}{2\mu_{D^*\overline{D}}} - k^5\right), \end{aligned} \tag{50}$$

Eq. (49) can be rewritten as

$$\begin{aligned} \mathcal{M}^{(2)}_{D^*\overline{D}} &= \frac{iC_1{}^2}{2(2\pi)^4}\int d^3k dk^5 \frac{1}{[\frac{1}{2(m_{D^*})}(-\mathbf{k}^2 - i\epsilon) - \Delta + k^5]} \\ &\times \frac{1}{[\frac{1}{2m_D}(\mathbf{k}^2 + i\epsilon) - \frac{\mathbf{p}^2}{2\mu_{D^*\overline{D}}} + k^5]}. \end{aligned} \tag{51}$$

where μ_{DD^*} is the reduced mass of the $D^*\overline{D}$ system,

$$\mu_{D^*\overline{D}} = \frac{m_{D^*}m_{\overline{D}}}{m_{D^*} + m_{\overline{D}}}, \tag{52}$$

and $\Delta = m_{D^*} - m_D$. The integral on k^5 can be performed with the residue theorem, giving finally the invariant amplitude in the form

$$\mathcal{M}^{(2)}_{D^*\overline{D}} = \frac{C_1{}^2}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\mathbf{k}^2 - 2\mu_{D^*\overline{D}}(E - \Delta) + i\epsilon}, \tag{53}$$

where $E = \mathbf{p}^2/2\mu_{D^*\overline{D}}$.

The results above obtained with manifestly Galilei-covariant XEFT for the scattering process $D^*\overline{D} \to D^*\overline{D}$ can be repeated in a similar manner for the scattering processes: $D\overline{D}^* \to D\overline{D}^*$, $D^*\overline{D} \to D\overline{D}^*$ and $D\overline{D}^* \to D^*\overline{D}$.

5 Bound States

In this section, we analyze the lowest-energy bound states which involve a pseudoscalar-heavy meson and a vector-heavy meson, which are eigenstates of charge conjugation, that is,

$$|X_\pm\rangle = \frac{1}{\sqrt{2}}\left[|D^*\overline{D}\rangle \pm |D\overline{D}^*\rangle\right]. \tag{54}$$

The motivation here follows Ref. [7], coming from the fact that if bound state exist, then the one connected to $|X_+\rangle$ can be associated to the $X(3872)$, since it has the correct quantum numbers $J^{PC} = 1^{++}$ [9, 44, 45]. On the other hand, a possible bound state related to $|X_-\rangle$ does not have the right quantum numbers of $X(3872)$, due to its odd nature under charge conjugation, which might be associated to another resonance.

Then, to determine the existence or not of bound-state with $J^{PC} = 1^{++}$, we must look for the poles of the transition amplitude $T_{++} = \langle X_+ |\mathbf{T}| X_+\rangle$. From the form of $|X_+\rangle$ given by Eq. (54), it is necessary to generalize the previous analysis in Section 4 by taking into account the coupled channel transition amplitude for the scattering processes $D^*\overline{D} \to D^*\overline{D}$, $D\overline{D}^* \to D\overline{D}^*$, $D^*\overline{D} \to D\overline{D}^*$ and $D\overline{D}^* \to D^*\overline{D}$, with the (leading-order) contact terms of the effective Lagrangians in Eqs. (30) and (31). Accordingly, the transition amplitude T_{++} is defined by

$$T_{++} = \langle X_+ |\mathbf{T}| X_+\rangle = \frac{1}{2}[T_{11} + T_{12} + T_{21} + T_{22}], \tag{55}$$

where the terms T_{11}, T_{12}, T_{21} and T_{22} are the transition amplitude elements of the matrix $\mathbf{T}$ written as

$$T_{11} = \left\langle D^*\overline{D}\right| \mathbf{T} \left|D^*\overline{D}\right\rangle, \tag{56}$$

$$T_{12} = \left\langle D^*\overline{D} |\mathbf{T}| D\overline{D}^*\right\rangle, \tag{57}$$

$$T_{21} = \left\langle D\overline{D}^* |\mathbf{T}| D^*\overline{D}\right\rangle, \tag{58}$$

$$T_{22} = \left\langle D\overline{D}^* |\mathbf{T}| D\overline{D}^*\right\rangle. \tag{59}$$

The computation of T_{++} elements in Eqs. (56)-(59) is performed similarly as in Sec. 4 and Ref. [7]. As a direct result, leading-order and next-to-leading-order terms are the contact and one-loop contributions, leading to $T_{ii} \sim C_1$ and $T_{ij,i\neq j} \sim C_2$, where C_1 and C_2 are the coupling constants defined in Eqs. (30) and (31). However, in order to obtain the complete next-to-leading-order terms for the transition amplitudes, we must also include other one-loop contributions engendered by effective Lagrangians. This sum is equivalent to the Lippmann-Schwinger equations [7]. In other words, the elements of the transition amplitude matrix in Eqs. (56)-(59) become the

Galilean covariant Bethe-Salpeter equations [46–49], being written as

$$iT_{11} = -iC_1 + \int \frac{d^5k}{(2\pi)^5} T_{11} G_{DD^*} C_1 \widetilde{\delta} - \int \frac{d^5k}{(2\pi)^5} T_{12} G_{DD^*} C_2 \widetilde{\delta}, \quad (60)$$

$$iT_{12} = iC_2 + \int \frac{d^5k}{(2\pi)^5} T_{12} G_{DD^*} C_1 \widetilde{\delta} - \int \frac{d^5k}{(2\pi)^5} T_{11} G_{DD^*} C_2 \widetilde{\delta}, \quad (61)$$

$$iT_{21} = iC_2 + \int \frac{d^5k}{(2\pi)^5} T_{21} G_{DD^*} C_1 \widetilde{\delta} - \int \frac{d^5k}{(2\pi)^5} T_{22} G_{DD^*} C_2 \widetilde{\delta}, \quad (62)$$

$$iT_{22} = -iC_1 + \int \frac{d^5k}{(2\pi)^5} T_{22} G_{DD^*} C_1 \widetilde{\delta} - \int \frac{d^5k}{(2\pi)^5} T_{21} G_{DD^*} C_2 \widetilde{\delta}. \quad (63)$$

Eqs. (60) to (63) can be given in the matrix form,

$$\begin{pmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{pmatrix} = \begin{pmatrix} -C_1 \\ C_2 \\ C_2 \\ -C_1 \end{pmatrix} + i\widetilde{\mathcal{M}} \begin{pmatrix} -C_1 & C_2 & 0 & 0 \\ C_2 & -C_1 & 0 & 0 \\ 0 & 0 & -C_1 & C_2 \\ 0 & 0 & C_2 & -C_1 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{pmatrix}, \quad (64)$$

where

$$\widetilde{\mathcal{M}} = \int \frac{d^5k}{(2\pi)^5} \, G_{DD^*} \, \widetilde{\delta}. \quad (65)$$

Therefore, the Bethe-Salpeter formalism [46–49] enables us to rewrite Eq. (55) as

$$T_{++} = \frac{\lambda}{1 - i\lambda\widetilde{\mathcal{M}}}, \quad (66)$$

where $\lambda = -C_1 + C_2$.

Notice that if the integral $\widetilde{\mathcal{M}}$ in Eq. (65) is manipulated analogously to the procedure performed from Eqs. (49) to (53), the result for the integral is in agreement with the quantity denoted as $\widetilde{A}$ in Ref. [7]. Thus, our Galilean formulation succeeds in reproducing the XEFT and heavy-quark symmetric effective theory outcomes.

We will spare the readers for the reproduction of procedure performed in Ref. [7]. Notwithstanding, for completeness we analyze the existence of bound states in a slightly different and more meticulous way. By applying the modified minimal subtraction scheme to Eq. (66), we achieve the renormalized transition amplitude,

$$T_{R++} = \frac{\lambda_R}{1 + \frac{i}{8\pi}\lambda_R \mu_{DD^*} \, |\mathbf{p}| \sqrt{1 - \frac{\Delta}{E}}}, \quad (67)$$

where the index R denotes the renormalized quantity. From Eq. (67) we remark that the renormalized coupling constants C_{1R} and C_{2R} (implicit in the definition of λ_R) play a fundamental role in determining the nature of the state $|X_+\rangle$. Therefore a necessary step is to characterize the dependence of T_{R++} with C_1 and C_2 and verify in what situations the findings are classified as bound states, resonances or virtual states. Correspondingly, we need to analyze the poles of the T-matrix. If we consider only s-wave bound states (which reproduce correctly the quantum numbers of $X(3872)$), the T-matrix can be written as

$$T_0 = \frac{1}{\frac{-1}{a_0} - ip}, \tag{68}$$

where a_0 is the scattering length and $p = |\mathbf{p}|$. The poles of the T-matrix in Eq. (68) are interpreted in the following way: (i) bound states: poles with $\Im p > 0$; (ii) resonances: poles with $\Re p > 0$ and $\Im p < 0$; and (iii) virtual states: poles corresponding to $\Im p > 0$. These poles are displayed in Fig. 1.

Accordingly, the position of the transition amplitude pole of the bound state on the energy scale obtained from Eq. (67) is

$$E_{Pole} = \frac{32\pi^2}{\lambda_R^2 \mu_{DD^*}^3} - \Delta. \tag{69}$$

Since the state $|X_+\rangle$ is a weakly-bound molecular state of the mesons $(D^*\overline{D} + c.c.)$, then its mass can be given as $M_X = (m_D + m_{D^*}) - E_b$, where E_b is the binding energy. Besides, keeping in mind that the position of this pole must be measured with respect to the constituent mass of the system, which in the present case is $2m_D$ [7, 29], then $M_X = 2m_D - E_{pole}$, which allows us to write E_b as

$$\begin{aligned} E_b &= (m_D + m_{D^*}) - M_X \\ &= \frac{32\pi^2}{\lambda_R^2 \mu_{DD^*}^3}. \end{aligned} \tag{70}$$

Also, from Eq. (68) we can get the scattering length,

$$a_0 = \frac{1}{\sqrt{2\mu_{DD^*} E_b}} = \frac{\lambda_R \mu_{DD^*}}{8\pi}. \tag{71}$$

From the development above, we can conclude that bound state solutions exist only if $\lambda_R > 0$. That is, the values for the parameters C_{1R} and C_{2R} must be such that $\lambda_R = -C_{1R} + C_{2R} > 0$.

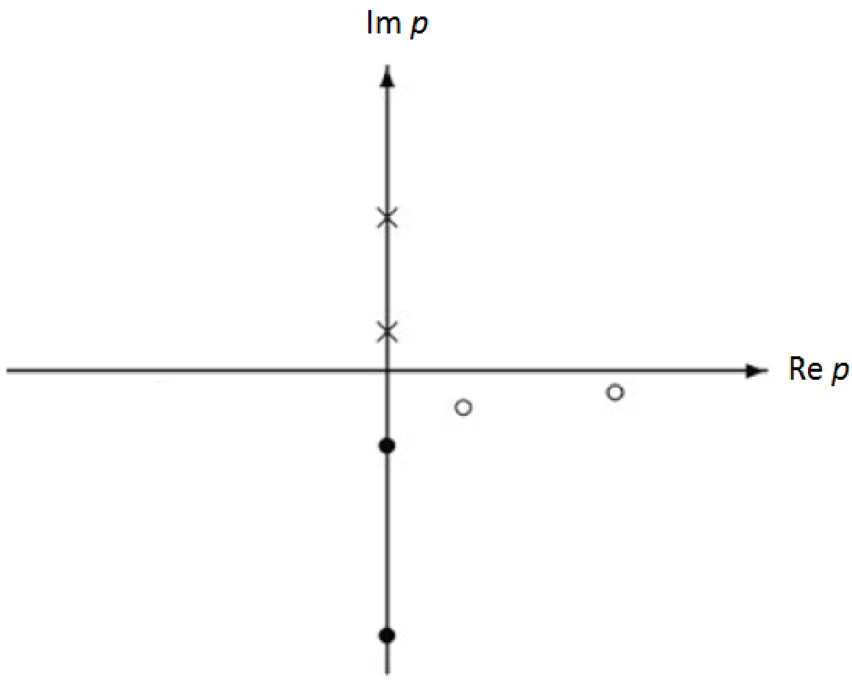

Figure 1: The pole distribution for the S-matrix in the complex plane: $\times$ represents bound states poles; $\circ$ poles corresponding to resonances, and $\bullet$ poles to virtual states.

Now we can update the results reported in Ref. [7]. By taking recent central values of the masses for neutral charmed mesons and $X(3872)$ from the Particle Data Group [4]: $m_{D^0} = 1864.84$ MeV, $m_{D^{*0}} = 2006.85$ MeV and for the $m_X = 3871.69$ MeV, respectively, then $E_b = 0$ MeV. It means that the neutral components does not engender binding, or it leads to a very loosely-bound state considering the error on the masses with $E_b = 0.2$ MeV. Nevertheless, with the discussion found in Ref. [15, 18], if one treats the $X(3872)$ as if it were dynamically generated also from the charged and strange charmed mesons, then the binding would be higher. For example, for the channel $(D^{*+}D^- + c.c.)$, with the masses $m_{D^-} = 1869.59$ MeV, $m_{D^{*+}} = 2010.26$ MeV, the binding energy becomes $E_b \simeq 8$ MeV.

Besides, it is interesting to emphasize the range of applicability of the present approach. As remarked in the previous section, we restrict our analysis to the region of relevance of contact-range interaction, that is the region where the pion-exchange contribution is not relevant. In this sense, a reasonable condition is to assume bound states which obey $a_0 \gtrsim 3\lambda_\pi$, where $\lambda_\pi = 1/m_\pi \sim 1$ fm is the pion Compton wavelength.

Thus, in Fig. 2 is shown the region of the allowed values for the constants C_{1R} and C_{2R} which generate bound states with E_b between 0.2 and 8 MeV in the range of validity of this framework. Note that the binding energy diminishes with λ_R, so that we see from this Figure that the lower limit of the allowed region is related to $E_b = 8$ MeV.

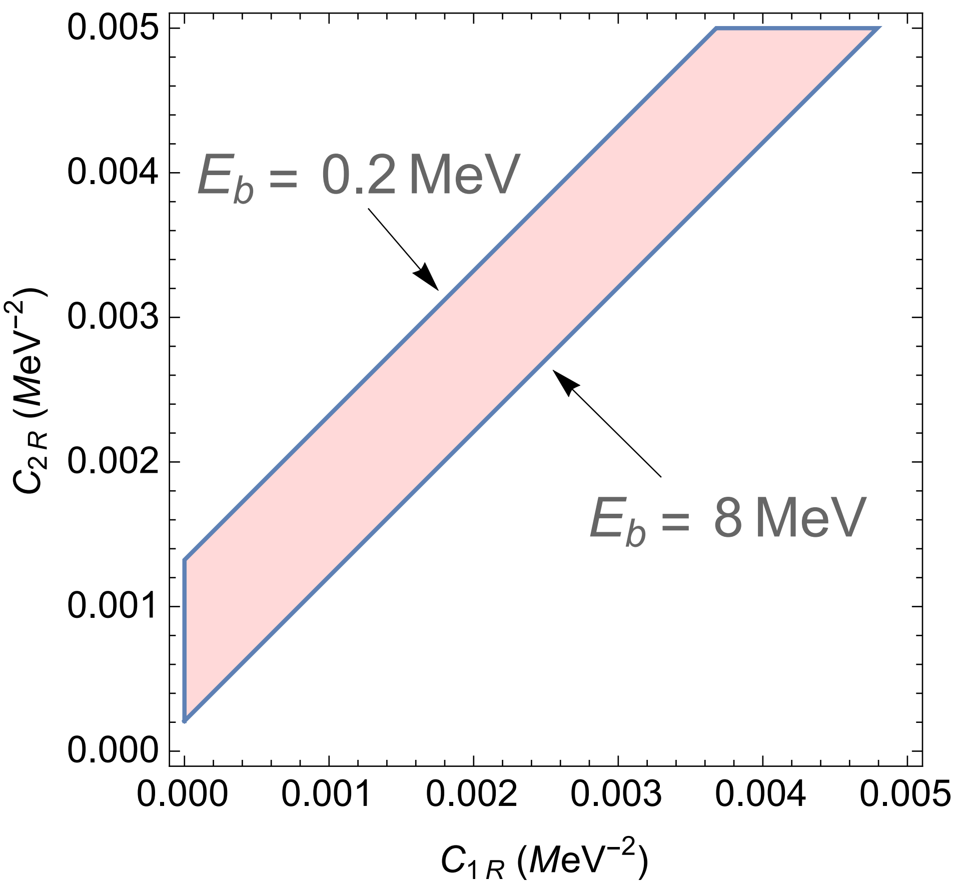

Figure 2: Region of the allowed values for the constants C_{1R} and C_{2R} which generate bound state for $|X_+\rangle$ with E_b between 0.2 and 8 MeV.

It is also worth mentioning at this point the orthogonal state to $|X_+\rangle$ with charge conjugation $C = -1$, defined in Eq. (54) by $|X_-\rangle$. Proceeding as before, the transition amplitude is [7]

$$\begin{aligned} T_{R--} &= \frac{1}{2}[T_{R11} - T_{R12} + T_{R21} - T_{R22}] \\ &= \frac{1}{2}[T_{11} - T_{12} + T_{21} - T_{22}] \\ &= \frac{\lambda'_R}{1 - i\lambda'_R \widetilde{\mathcal{M}}_R}, \end{aligned} \tag{72}$$

where $\lambda'_R = -C_{R1} - C_{R2}$. We see that in the region of molecular state, $|X_+\rangle$ does not produce bound state for $|X_-\rangle$, since $\lambda'_R < 0$.

We complete the analysis by referring to the heavy-meson molecule $(B\overline{B}^* + c.c.)$, where $B^{(*)}$ denote the bottom mesons. We consider in this case only neutral bottom mesons, with the following masses [4]: $m_{B^0} = 5279.63$ MeV and $m_{B^{*0}} = 5324.65$ MeV. This case is done as the procedure above, with binding energy being given by Eq. (70), and using the appropriate replacement of reduced mass $\mu_{DD^*} \to \mu_{BB^*}$. So, the bound states are analyzed as in the previous case, with E_b being restricted from 0.2 MeV up to the value respecting the condition $a_0 \gtrsim 3\lambda_\pi$. Fig. 3 shows the region of the allowed values for the constants C_{1R} and C_{2R} which generate bound states with $J^{PC} = 1^{++}$ for $(B\overline{B}^* + c.c.)$, taking E_b in the range of validity of this framework.

6 Concluding remarks

We have constructed the Galilei-covariant version of an effective theory containing non-relativistic heavy mesons and pions as degrees of freedom. We have made use of an appropriate five-dimensional manifold to describe covariant non-relativistic physics. The requirement of Galilean covariance has yielded effective Lagrangians without ambiguities.

As an application, the leading-order scattering amplitudes and the properties of possible heavy-meson bound states have been calculated and discussed. In particular, heavy-meson molecules with $J^{PC} = 1^{++}$ for the states $(D\overline{D}^* + c.c.)$ and $(B\overline{B}^* + c.c.)$ have been analyzed, taking care of the range of validity of this framework. Our findings have demonstrated the success of Galilean formulation in reproducing the XEFT and heavy-quark symmetric effective theory outcomes.

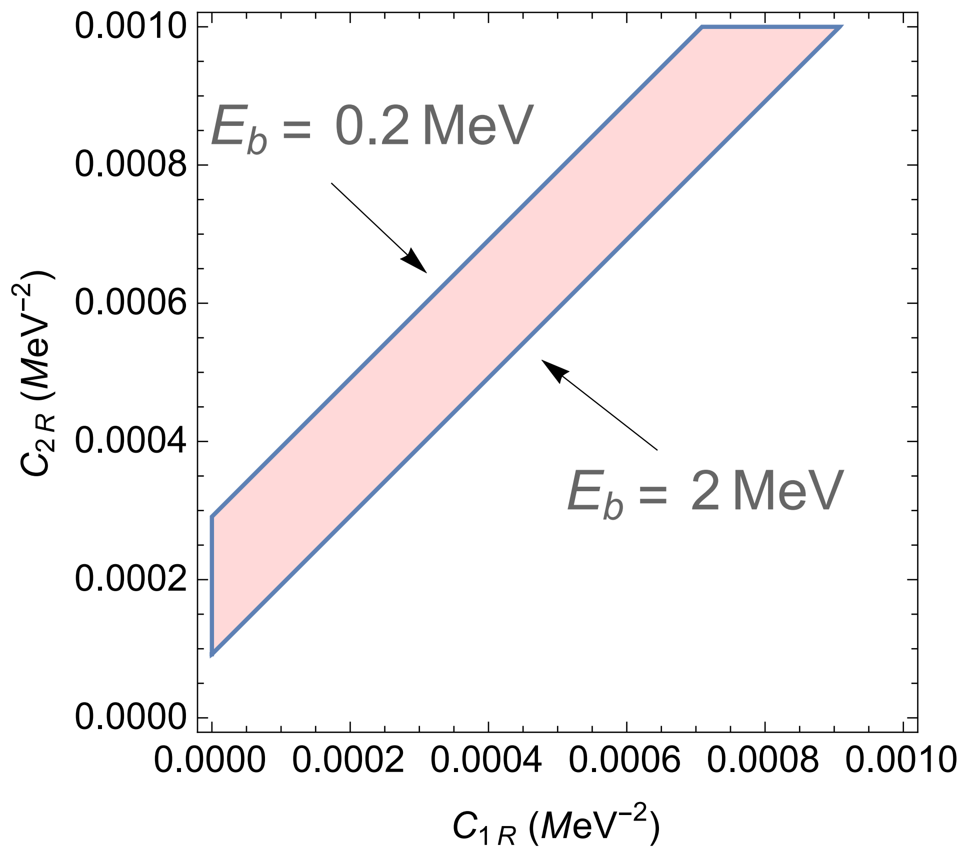

Figure 3: Region of the allowed values for the constants C_{1R} and C_{2R} which generate bound state with $J^{PC} = 1^{++}$ for $(B\overline{B}^* + c.c.)$, assuming E_b in the range of validity of this framework.

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